Surfing instability on a viscous shallow fluid

1990 J. Phys. A: Math. Gen. 23 L1101

(http://iopscience.iop.org/0305-4470/23/21/005)

J. Phys. A: Math. Gen. 23 (1990) L1101–L1104. Printed in the UK

LETTER TO THE EDITOR

Surfing instability on a viscous shallow fluid

M Papoular

Centre de Recherche sur les Très Basses Températures, CNRS, BP 166, Av des Martyrs, 38042 Grenoble, France

Received 3 September 1990

Abstract. We argue that a viscous fluid layer, if sufficiently shallow, will deform under the drag exerted by a thin, rigid plate sliding upon it. The resulting gravity wave travels at the plate velocity.

Consider a thin, light and rigid plate floating upon a horizontal layer of a viscous, heavier fluid. Let us ignore end effects by taking infinite dimensions in the horizontal plane: the only geometrical length is the layer thickness h. Set the plate into uniform motion at speed V. We ask: is there a characteristic depth h_c—and a related characteristic velocity V_c—marking the threshold of an instability behaviour in the fluid?

It is natural to expect such an instability to take the form of a 'shallow-water' *gravity wave* accompanying the plate translation (figure 1). It is well known [1] that a long (wavelength $\lambda \gg$ depth h) gravity wave propagates, with no dispersion, at velocity:

$$V = \sqrt{gh} \qquad \lambda \gg h \tag{1}$$

where g is gravity acceleration. This result implies a linearity condition: $A \ll h$ on the amplitude A of the wave. We have taken the same symbol V for wave velocity and speed of plate since equality of these will favour build-up of the instability. For the sake of simplicity we henceforth assume this 'resonance condition' to be fulfilled.

Figure 1 shows the instability to take the plate upwards over an extra height equal to the wave amplitude A. We consider the plate to be so light that the corresponding potential energy can be neglected.

A more serious source of potential energy brought in along with the instability is the creation of a *free surface*. This is just given, per unit area, by $\Delta\sigma$, the corresponding increase in total surface energy. That is the price to be paid in order for the instability to build up and reduce the viscous dissipation as we shall see. And, as such, it may well entail metastable behaviour at onset of instability—but we shall not consider here this aspect of the problem.

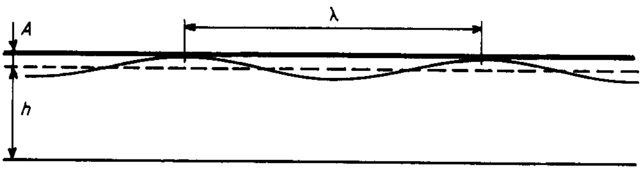

Figure 1. Scheme of the fluid-layer instability: gravity wave with amplitude A, wavelength and velocity $V = \sqrt{gh}$.

0305-4470/90/211101+04$03.50

There are a number of conditions to be placed on the fluid viscosity η. First, the gravity wave should not be heavily damped. For ordinary gravity waves, with frequency $\omega = \sqrt{gq}$, this takes on the simple form [1]:

$$\frac{\eta}{\rho} q^2 \ll \sqrt{gq} \tag{2}$$

with ρ the fluid specific mass, and $q = 2\pi/\lambda$ the wavevector. For the centimetric or decimetric gravity waves we are interested in, this is obviously a very mild condition. Also, note that for these wavelengths we can ignore the capillary (ripplon) contribution:

$$\frac{\sigma q^3}{\rho} \ll gq \tag{3}$$

for typical values of surface tension σ.

A second condition on viscosity η is that, if we want to stay in a simple regime, the viscous penetration depth (at the wave frequency ω) should be small on the scale of the layer thickness:

$$\delta = \sqrt{\frac{2\eta}{\rho\omega}} \ll h. \tag{4}$$

With $\omega = Vq = 10\ \mathrm{s}^{-1}$ ($q = 2\pi/\lambda = 0.1\ \mathrm{cm}^{-1}$, $V = 100\ \mathrm{cm\ s}^{-1}$), $\rho = 2\ \mathrm{g\ cm}^{-3}$ and $\eta = 0.1$ poise, we get $\delta = 0.1$ cm. So this, again, is not a very restrictive condition. Note that, given (4), (2) is automatically satisfied, in view of (1).

Viscosity, on the other hand, should be *large enough* (within condition (4)) so that dissipation after the onset of instability is *lower* than before. This is our central point. It can be viewed as a form of the 'principle' of minimum entropy production (which is known to fail for systems driven too far away off equilibrium, but we have already insisted that A stay much smaller than h). Let us now develop this point.

Before instability onset, the energy dissipated per unit time and unit area is given by:

$$\dot{E}_b = \frac{1}{2}\eta\frac{V^2}{h}. \tag{5}$$

This is the standard result for viscous Stokes flow (uniform shear gradient taking the velocity from V to zero over layer thickness h).

To find the energy dissipated after onset is a slightly more complicated exercise. Now E is given by [1, p 100]:

$$\dot{E}_a = \frac{v_0^2}{2}\sqrt{\frac{\eta\rho\omega}{2}} = \frac{v_0^2}{2}\sqrt{\frac{\pi\eta\rho V}{\lambda}} < \dot{E}_b \tag{6}$$

v_0 being the amplitude of the velocity oscillation in the gravity wave. This formula, with its characteristic square-root group, results from the fact that the main source of viscous dissipation now resides in the narrow rotational-flow sublayer of width δ (equation 4) at the bottom of the fluid. The flow-velocity v which, at a given abscissa, is almost uniform along the depth of the fluid layer, obeys Euler's equation:

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \tag{7}$$

(ζ: instantaneous vertical coordinate of free surface; we have neglected viscosity since the wave is not appreciably damped). Since $\omega/q = V$, we may rewrite (7) as

$$v_o V = gA \tag{8}$$

where A is the amplitude of our long gravity wave; recall that $A \ll h$. Comparing with (1), we get:

$$A = h\frac{v_o}{V}. \tag{9}$$

So, $v_o \ll V$: the flow velocity in the wave has to be much smaller than the phase velocity (i.e. the plate velocity), if we are to stay in the linear, small amplitude regime. This is a somewhat delicate point since the no-slip condition must hold at the contact with the plate, as it does at the bottom wall of the vessel. Clearly, the ratio v_o/V (and, therefore, A/h) should have an optimal value, below which the drag exerted by the plate on the gravity wave is not efficient, and above which nonlinearities would prohibitively increase dissipation. This optimal value might depend on material parameters such as η and h (and therefore, $V = \sqrt{gh}$), as well as the wavelength λ. In this letter we shall limit ourselves to a simple, *heuristic* criterion:

$$\frac{A}{h} = \frac{v_o}{V} = 0.1. \tag{10}$$

(In fact, as we shall see in the next section, A must indeed be larger than the bending of the plate.)

Inserting (10), and (1), in the dissipation condition (6) ($\dot{E}_a < \dot{E}_b$), gives:

$$\eta\lambda h^{-5/2} > 10^{-4}\pi\rho\sqrt{g}. \tag{11}$$

Take, as an example: $h = 10$ cm (so, $V \simeq 100$ cm s^{-1} and $A = 1$ cm), $\eta = 0.1$ poise, $\rho = 3$ g cm^{-3}. The minimum unstable wavelength is then $\lambda = 100$ cm.

Besides (11), and the trivial long-wave condition $\lambda \gg h$, there is another inequality condition between h and λ which must be satisfied in order for the instability to set in: the amplitude of the wave, A, must be *larger than the bending amplitude of the plate* (not represented on figure 1); otherwise contact is maintained everywhere and there is no point for the instability to take place. We write therefore:

$$A\left(\approx\frac{h}{10}\right) > B \simeq \left(\frac{\lambda^4\rho_p g}{E}\right)^{1/3} \tag{12}$$

where, following [2], we have expressed the bending B in terms of λ, g, ρ_p and $\mathscr{E}$ (respectively, specific mass and Young modulus of the plate; $\rho_p g$ is the weight per unit volume of the plate). This formula implies the plate thickness to be much smaller than the bending B. So, now, the wavelength λ must be smaller than $h^{3/4}$ times a constant involving the material parameters ρ_p and $\mathscr{E}$, and gravity. With $h = 10$ cm, $\rho_p = 1$ g cm^{-3} and $\mathscr{E} = 10^{11}$ dy cm^{-2} (a stiff plate), inequality (12) amounts to: $\lambda_{max} \simeq$ 100 cm, just matching condition (11) (with the above choice of parameters).

Conditions (11) and (12), when expressed on a graph (λ against h) delimit an instability region: see figure 2 (the milder condition (4) has not been represented on this figure).

To conclude, we have described a 'surfing' instability for a thin, light and rigid plate sliding upon a shallow, viscous fluid layer. For every set of material parameters, there is, in the (h, λ) plane, a limiting point (point C in figure 2) below which (i.e.

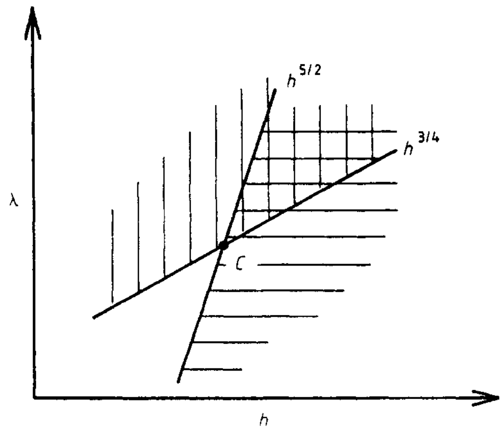

Figure 2. Instability region: lower, unhatched area. With material parameters chosen as in text, $h_c = 10$ cm ($V_c = 100$ cm s^{-1}, $A = 1$ cm) and $\lambda_c = 100$ cm.

for smaller thicknesses h and shorter wavelengths λ) the instability is favoured. Point C moves upwards, to larger h and λ, as the viscosity increases. We have mentioned, without detailed discussion, possible metastabilities associated with the creation of a free surface, and possibly more complicated wave-amplitude to layer-thickness ratios than given in (10).

The intrinsic difficulty of the wind-wave problem is well known. The 'board-wave' instability, at least in its minimal version as presented here, seems to be understandable in simpler terms: roughly speaking, the board tends to create the wave in order to improve sliding efficiency. We realize that other, more elaborate treatments probably exist in the literature. We have bluntly left aside a number of 'realistic' parameters, likely to affect the problem to varying degrees: wetting, viscoelasticity, end effects, profile effects, etc. We think however that this instability, as we have described it, might show up in various problems in physical hydrodynamics or industrial flows.

References

[1] Landau L and Lifshitz E 1959 *Fluid Mechanics* (New York: Pergamon)
[2] Landau L and Lifshitz E 1957 *Théorie de l'Elasticité* (Moscow: Mir)